STARS

University of Central Florida
STARS

Faculty Bibliography 1990s | Faculty Bibliography

1-1-1995

The Painleve Formulations And Exact-Solutions Of The Nonlinear Evolution-Equations For Modulated Gravity-Wave Trains

Bhimsen K. Shivamoggi
University of Central Florida

David K. Rollins
University of Central Florida

Find similar works at: https://stars.library.ucf.edu/facultybib1990
University of Central Florida Libraries http://library.ucf.edu

This Article is brought to you for free and open access by the Faculty Bibliography at STARS. It has been accepted for inclusion in Faculty Bibliography 1990s by an authorized administrator of STARS. For more information, please contact STARS@ucf.edu.

Recommended Citation

Shivamoggi, Bhimsen K. and Rollins, David K., "The Painleve Formulations And Exact-Solutions Of The Nonlinear Evolution-Equations For Modulated Gravity-Wave Trains" (1995). *Faculty Bibliography 1990s*. 1177.
https://stars.library.ucf.edu/facultybib1990/1177

University of Central Florida
STARS
Showcase of Text, Archives, Research & Scholarship

Journal of Mathematical Physics

The Painlevé formulations and exact solutions of the nonlinear evolution equations for modulated gravity wave trains

Cite as: Journal of Mathematical Physics **35**, 4779 (1994); https://doi.org/10.1063/1.530814
Submitted: 31 March 1993 . Accepted: 29 March 1994 . Published Online: 04 June 1998

Bhimsen K. Shivamoggi, and David K. Rollins

View Online

Export Citation

ARTICLES YOU MAY BE INTERESTED IN

Beamspace compressive spatial spectrum estimation on large aperture acoustic arrays
The Journal of the Acoustical Society of America **136**, 2214 (2014); https://doi.org/10.1121/1.4900033

The dependence of solar energetic particle fluences on suprathermal seed-particle densities
AIP Conference Proceedings **1500**, 128 (2012); https://doi.org/10.1063/1.4768755

Some medical applications of acoustic streaming
The Journal of the Acoustical Society of America **117**, 2530 (2005); https://doi.org/10.1121/1.4788397

© 1994 American Institute of Physics.

The Painlevé formulations and exact solutions of the nonlinear evolution equations for modulated gravity wave trains

Bhimsen K. Shivamoggi and David K. Rollins
University of Central Florida, Orlando, Florida 32816

(Received 31 March 1993; accepted for publication 29 March 1994)

In this article the integrability aspects of the nonlinear evolution equations for modulated gravity wave trains are investigated by demanding the Painlevé property of the solutions. The Painlevé formulations also lead to kink-shaped exact localized solutions of these equations.

I. INTRODUCTION

Zakharov[1] showed that the evolution of a weakly nonlinear, nearly monochromatic deep-water gravity wave train subjected to a two-dimensional modulation is governed by a nonlinear Schrödinger equation for the modulated complex envelope A

$$2i\omega\left(\frac{\partial A}{\partial t}+\omega'\,\frac{\partial A}{\partial x}\right)+\omega\omega''\,\frac{\partial^2 A}{\partial x^2}+\frac{\omega\omega'}{k}\,\frac{\partial^2 A}{\partial y^2}-\omega_0^2 k_0^2|A|^2A=0, \tag{1}$$

where ω_0 and k_0 are the frequency and the wave number, respectively, of the carrier wave propagating in the x direction, and the primes on ω denote differentiation with respect to k. Benney and Roskes[2] gave a rigorous derivation of Eq. (1), using the method of multiple scales.

The properties of Eq. (1) are remarkably different from those of its one-dimensional counterpart. The unbounded nature of the linear instability region in the wave number space for Eq. (1) is one such feature. Yuen and Fergusson,[3] and Martin and Yuen[4] numerically solved Eq. (1) for spatially periodic boundary conditions and showed that the long-time evolution of the linearly unstable solution is composed of the growth and decay of all the harmonics of the initial perturbation that lie within the unstable region, each one alternately dominating the evolution. Martin and Yuen[4] found that the energy initially contained in low wave number modes leaked to linearly unstable arbitrarily higher harmonics so that, for long times, the energy sharing occurred among the arbitrarily high wave number modes. Shivamoggi and Mohapatra[5] gave an analytic explanation of this phenomenon using the ideas advanced earlier by Thyagaraja[6] for the one-dimensional counterpart of Eq. (1). This was accomplished by showing that the nonlinear wave system associated with Eq. (1) can be described as effectively possessing an arbitrarily high number of degrees of freedom so that the overall motion can at best be only quasirecurrent.[5]

The effect of finite depth h of the water on the evolution of the gravity wave train was considered by Davey and Stewartson[7] and Djordjevic and Redekopp.[8] In water of finite depth, the gravity wave train induces a mean flow, which is a nonlocal effect. The evolution of this gravity wave train is then described by a nonlinear Schrödinger equation like Eq. (1) coupled to a Poisson-type equation for the potential Φ of the mean flow

$$2i\omega\left(\frac{\partial A}{\partial t}+\omega'\,\frac{\partial A}{\partial x}\right)+\omega\omega''\,\frac{\partial^2 A}{\partial x^2}+\frac{\omega\omega'}{k}\,\frac{\partial^2 A}{\partial y^2}=\omega_0^2k_0^2\left[\frac{(1-\sigma^2)(9-\sigma^2)}{\sigma^4}+8-\frac{2(1-\sigma^2)^2}{\sigma^2}\right]|A|^2A$$
$$+k_0\left[1+\frac{(1-\sigma^2)}{4}\right]A\,\frac{\partial\Phi}{\partial x}, \tag{2}$$

 © 1994 American Institute of Physics

$$\left(1-\frac{\omega'^2}{gh}\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=-\frac{\omega_0}{2\sigma h}\left[1+\frac{(1-\sigma^2)}{4}\right]\frac{\partial}{\partial x}|A|^2, \tag{3}$$

where $\sigma=\tanh k_0 h$.

Several investigations have been made of the properties of Eqs. (2) and (3) (Freeman and Davey,[9] Anker and Freeman,[10] Ablowitz and Haberman,[11] Ablowitz and Segur,[12] Tajiri and Hagiwara,[13] etc.). An inverse scattering transform (Ablowitz and Haberman[11]) and soliton solutions (Anker and Freeman[10]) of Eqs. (2) and (3), in the long wave limit, have been given. Ablowitz and Segur[12] conjectured that Eqs. (2) and (3) are not integrable over the entire range of parameters.

It is now well known that the integrability of a dynamical system is intimately linked with its analytical structure. By using the Painlevé property (Ince[14]) of the solutions, namely, that the only movable singularities are poles, we will investigate in this article whether Eq. (1) and the system of Eqs. (2) and (3) are integrable. [It is a well-known fact that the one-dimensional counterpart of Eq. (1) is integrable (Ablowitz and Segur[15]).] For this purpose, it proves to be convenient to establish first the Lie group symmetries of Eq. (1) and the system of Eqs. (2) and (3). The Painlevé formulation, interestingly enough, also leads to kink-shaped exact solutions of Eq. (1) and Eqs. (2) and (3) which are localized about a certain line in the x,y plane.

II. NONLINEAR EVOLUTION EQUATION FOR MODULATED GRAVITY WAVE TRAINS IN DEEP WATER

In order to investigate the analytic properties of Eq. (1), let us first rewrite it in the following form:

$$iu_3-u_{11}+u_{22}+\mu u^2u^*=0, \tag{4}$$

where $x_1=x$, $x_2=y$, and $x_3=t$ and the subscript j on u denotes differentiation with respect to x_j. Here, μ is a real interaction parameter ($\mu<0$ for nonlinear gravity waves) and u^* is the complex conjugate of u.

A. Conservation laws

It may be readily verified that the two-dimensional nonlinear Schrödinger equation (4) has a variational characterization. It corresponds to extremizing the action integral

$$J(u)=\int\int\int L(x_1,x_2,x_3,u_1,u_2,u_3)dx_1\,dx_2\,dx_3, \tag{5}$$

where the Lagrangian density L is given by

$$L=\frac{i}{2}(u^*u_3-uu_3^*)+|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4. \tag{6}$$

Conservation laws can be found by applying Noether's theorem to the functional $J(u)$. $J(u)$ is invariant under translations in x_1, x_2, and x_3 and the scaling $u \Rightarrow ue^{i\epsilon}$. Thus, under a local Lie group of transformations

$$\bar{x}_\alpha=x_\alpha+\epsilon\xi^{(\alpha)}+O(\epsilon^2),\quad \bar{u}=u+\epsilon\eta+O(\epsilon^2) \tag{7}$$

defined by infinitesimal generators $\xi^{(\alpha)}$, $\alpha=1,2,3$ and η, we have the following conservation laws (Logan[16]):

$$\frac{\partial}{\partial x_\alpha}\left[L\xi^{(\alpha)}+\frac{\partial L}{\partial u_\alpha}(\eta-u_\beta\xi^{(\beta)})+\frac{\partial L}{\partial u_\alpha^*}(\eta^*-u_\beta^*\xi^{(\beta)})\right]=0, \tag{8}$$

where repeated indices are summed over.

For the scaling in u, $\xi^{(\alpha)}=0$ for $\alpha=1,2,3$ and $\eta=iu$, and Eq. (8) becomes

$$\frac{\partial}{\partial x_1}(iu^*u_1-iuu_1^*)+\frac{\partial}{\partial x_2}(iuu_2^*-iu^*u_2)+\frac{\partial}{\partial x_3}(|u|^2)=0, \tag{9}$$

which shows that $|u|^2$ is an invariant.

For a translation in x_1, $\xi^{(1)}=1$ with $\xi^{(2)}=\xi^{(3)}=\eta=0$, and Eq. (8) becomes

$$\frac{\partial}{\partial x_1}\left(\frac{i}{2}(u^*u_3-uu_3^*)-|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4\right)+\frac{\partial}{\partial x_2}(u_1u_2^*+u_1^*u_2)$$
$$+\frac{\partial}{\partial x_3}\left(\frac{i}{2}(uu_1^*-u^*u_1)\right)=0. \tag{10}$$

Similarly, for a translation in x_2, $\xi^{(2)}=1$, $\xi^{(1)}=\xi^{(3)}=\eta=0$, and Eq. (8) becomes

$$\frac{\partial}{\partial x_1}(-u_1u_2^*-u_1^*u_2)+\frac{\partial}{\partial x_2}\left(\frac{i}{2}(u^*u_3-uu_3^*)+|u_1|^2+|u_2|^2+\frac{\mu}{2}|u|^4\right)$$
$$+\frac{\partial}{\partial x_3}\left(\frac{i}{2}(uu_2^*-u^*u_2)\right)=0. \tag{11}$$

For a translation in x_3, $\xi^{(3)}=1$, $\xi^{(1)}=\xi^{(2)}=\eta=0$, and Eq. (8) becomes

$$\frac{\partial}{\partial x_1}(-u_1^*u_3-u_1u_3^*)+\frac{\partial}{\partial x_2}(u_2^*u_3+u_2u_3^*)+\frac{\partial}{\partial x_2}(u_2^*u_3+u_2u_3^*)$$
$$+\frac{\partial}{\partial x_3}\left(|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4\right)=0, \tag{12}$$

which leads to the "energy" invariant

$$\int\int\left(|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4\right)dx_1\,dx_2=\text{const.} \tag{13}$$

Observe that the "energy" for the two-dimensional nonlinear Schrödinger equation (4) has an indefinite character in marked contrast to the one-dimensional version of Eq. (4). Thus, the motion of the effective "particle" represented by this invariant can become unbounded. Physically, this is due to the fact that the transverse component of the modulation behaves as a negative-energy mode (Cairns[17]).

B. The Lie group symmetries

The problem of determining the full symmetry group which leaves a given equation invariant is generally a difficult one in practice. One therefore looks for infinitesimal symmetry groups. Typically, the latter type of symmetries may be used to introduce the so-called similarity variables which may in turn be used to reduce the number of independent variables. In the case of integrable systems, such a reduction may lead to one of the Painlevé transcendents with suitable transformations. Let us consider the invariance of Eq. (4) under a local Lie group of infinitesimal transformations with generators $\xi^{(j)}$, $j=1,2,3$ and η

$$\begin{aligned}\bar{u}&=u+\epsilon\eta(x_1,x_2,x_3,u,u^*)+O(\epsilon^2),\\ \bar{x}_j&=x_j+\epsilon\xi^{(j)}(x_1,x_2,x_3,u,u^*)+O(\epsilon^2),\quad j=1,2,3.\end{aligned}\tag{14}$$

This then yields the following condition on $\xi^{(j)}$ and η (and its extensions $\eta^{(k)}$):

$$i\eta_3^{(1)}-\eta_{11}^{(2)}+\eta_{22}^{(2)}+2\mu uu^*\eta+\mu u^2\eta^*=0,\tag{15}$$

where the extensions $\eta^{(k)}$ are given by

$$\begin{aligned}\eta_j^{(1)}&=\eta_j+\eta_u u_j-\xi_j^{(l)}u_l,\\ \eta_{11}^{(2)}&=\eta_{11}+2\eta_{1u}u_1+\eta_u u_{11}-\xi_{11}^{(l)}u_l-2\xi_1^{(l)}u_{1l},\\ \eta_{22}^{(2)}&=\eta_{22}+2\eta_{2u}u_2+\eta_u u_{22}-\xi_{22}^{(l)}u_l-2\xi_2^{(l)}u_{2l},\end{aligned}\tag{16}$$

and we have used the result (see Bluman and Kumei[18]) that, for Eq. (4) which is of a quasilinear type, the generators $\xi^{(j)}$ and η satisfy the following conditions:

$$\left.\begin{aligned}&\frac{\partial\xi^{(j)}}{\partial u}=0;\quad j=1,2,3,\\ &\frac{\partial^2\eta}{\partial u^2}=0.\end{aligned}\right\}\tag{17}$$

In Eq. (16), summation over repeated index l is implied.

Using Eqs. (16) and (17), and following the standard procedure to obtain the symmetry group generators, we have

$$\left.\begin{aligned}&\xi^{(1)}=(\alpha x_3+\beta)x_1+\gamma x_2+\delta x_3+\sigma,\\ &\xi^{(2)}=\gamma x_1+(\alpha x_3+\beta)x_2+\Omega x_3+\kappa,\\ &\xi^{(3)}=2(\tfrac{1}{2}\alpha x_3^2+\beta x_3)+\chi,\\ &\eta=\left[-\frac{i}{2}\left(\frac{1}{2}\alpha x_1^2+\delta x_1\right)+\frac{i}{2}\left(\frac{1}{2}\alpha x_2^2+\Omega x_2\right)-(\alpha x_3+\beta-i\omega)\right]u.\end{aligned}\right\}\tag{18}$$

Equation (18) gives a nine-parameter local Lie group of transformations admitted by Eq. (4) with infinitesimal generators given by

$$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},\quad X_3=\frac{\partial}{\partial x_3},$$

$$X_4=x_1x_3\frac{\partial}{\partial x_1}+x_2x_3\frac{\partial}{\partial x_2}+x_3^2\frac{\partial}{\partial x_3}-\left(x_3+\frac{ix_1^2}{4}-\frac{ix_2^2}{4}\right)u\frac{\partial}{\partial u},$$

$$X_5=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+2x_3\frac{\partial}{\partial x_3}-u\frac{\partial}{\partial u},\quad X_6=x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2},\tag{19}$$

$$X_7=x_3\frac{\partial}{\partial x_1}-\frac{i}{2}x_1u\frac{\partial}{\partial u},\quad X_8=x_3\frac{\partial}{\partial x_2}+\frac{i}{2}x_2u\frac{\partial}{\partial u},\quad X_9=iu\frac{\partial}{\partial u}.$$

X_1, X_2, and X_3 merely reflect the invariance of Eq. (4) under translations in t, x, and y. X_9 corresponds to a gauge transformation $u \Rightarrow ue^{i\epsilon}$. X_4, X_5, X_6, X_7, and X_8 correspond to scaling invariances of Eq. (4). The groups corresponding to X_2, X_4, X_6, and X_8 are peculiar to the two-dimensional nonlinear Schrödinger equation (4) and do not reduce to the Lie group of the one-dimensional counterpart of Eq. (4) in the appropriate limit. It should be mentioned that a symmetry group similar to Eq. (18) was given earlier by Tajiri[19] for a general two-dimensional nonlinear Schrödinger equation.

C. The Painlevé formulation

1. Symmetry group corresponding to the generator X_4

Consider the symmetry group corresponding to the parameter α. The invariant solutions corresponding to X_4 satisfy

$$\frac{dx_1}{x_1x_3}=\frac{dx_2}{x_2x_3}=\frac{dx_3}{x_3^2}=-\frac{du}{(x_3+(ix_1^2/4)-(ix_2^2/4))u}\tag{20}$$

and so have the form

$$u=\frac{1}{x_3}e^{(i/4x_3)(x_2^2-x_1^2)}F(\psi,\phi),\tag{21}$$

where

$$\psi=\frac{x_1}{x_3},\quad \phi=\frac{x_2}{x_3}.\tag{22}$$

Using Eq. (21), Eq. (4) becomes

$$F_{\phi\phi}-F_{\psi\psi}+\mu F^2F^*=0.\tag{23}$$

The form of Eq. (23) suggests a further scaling transformation

$$F=\frac{1}{\psi}G(s),\quad s=\frac{\psi}{\phi}.\tag{24}$$

One may readily verify that Eq. (24) merely corresponds to the symmetry group associated with the infinitesimal generator X_5! Using Eq. (24), Eq. (23) becomes

$$-\frac{2}{s^4}G+2\left(\frac{1}{s^3}+\frac{1}{s}\right)G'-\left(\frac{1}{s^2}-1\right)G''+\frac{\mu}{s^4}G^2G^*=0, \tag{25}$$

where primes denote differentiation with respect to s.

In order to investigate the Painlevé property of the solutions of Eq. (25), it is necessary to first remove the nonanalytic nature of Eq. (25). For this purpose, one usually embeds Eq. (25) in a suitable coupled system (Ablowitz and Segur[15]). However, we will proceed differently here, and put, instead

$$G(s)=H(s)e^{i\theta(s)}, \tag{26}$$

where H and θ are real, and separate the resulting equation into real and imaginary parts

$$-2H+2(s+s^3)H'+(s^4-s^2)H''-(s^4-s^2)H\theta'^2+\mu H^3=0, \tag{27}$$

$$2(1+s^2)H\theta'+2(s^3-s)H'\theta'+(s^3-s)H\theta''=0. \tag{28}$$

The Painlevé property can be proven in a straightforward manner by expanding the solutions into a Laurent series and investigating whether these expansions contain sufficient number of arbitrary constants to cover the entire solution manifold of Eqs. (27) and (28).

The leading behavior of solutions of Eqs. (27) and (28) at a movable singularity s_0 is determined by substituting

$$H=az^{\alpha_1}, \quad \theta=bz^{\alpha_2}, \quad z\equiv s-s_0 \tag{29}$$

into Eqs. (27) and (28), and balancing the most singular terms, to obtain

$$\left.\begin{array}{l} a^2=-\dfrac{2(s_0^4-s_0^2)}{\mu}, \quad \alpha_1=-1, \quad \alpha_2=0, \\ b \ \text{arbitrary.} \end{array}\right\} \tag{30}$$

The Painlevé analysis consists in determining the behavior in the neighborhood of the singularity s_0, by constructing local expansions with the above leading behaviors as leading terms. Equations (27) and (28) are said to exhibit the Painlevé property if these expansions have a simple Laurent series so that the singularity is indeed a movable pole. The powers of z at which the arbitrary coefficients appear in the series, i.e., the resonances, are determined by setting

$$H=az^{-1}+pz^{-1+\sigma} \tag{31}$$

and balancing the most singular terms again in Eqs. (27) and (28). This gives

$$\sigma=-1,4 \tag{32}$$

in order that p is arbitrary. The root $\sigma=-1$, as usual, corresponds to the arbitrariness of s.

We thus consider the expansions

$$\left.\begin{array}{l} H=az^{-1}+a_0+a_1z+a_2z^2+a_3z^3+\cdots, \\ \theta=b+b_1z+b_2z^2+b_3z^3+b_4z^4+\cdots. \end{array}\right\} \tag{33}$$

Substituting Eq. (33) into Eqs. (27) and (28), and collecting terms of equal powers of z, we obtain the recursion relations for the a_j's and b_j's

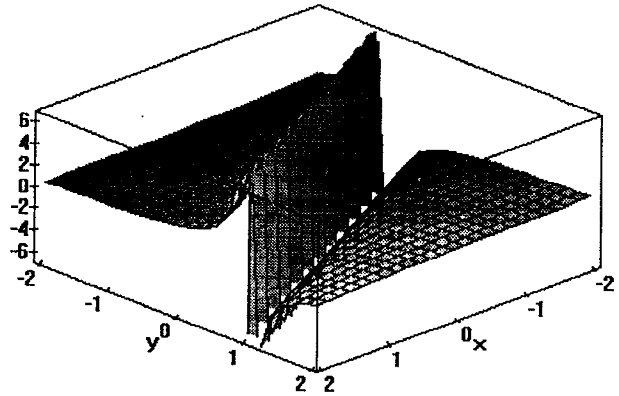

FIG. 1. Plot of $|u|$ as given in Eq. (36) with parameter values $b=0$, $c=2$, and $\mu=-1$.

$$\begin{gathered} a^2=-\frac{2(s_0^4-s_0^2)}{\mu}, \quad b \text{ arbitrary}, \\ a_0=\frac{a}{s_0}, \quad b_1=0, \quad a_1=0, \quad b_2=0, \quad a_2=0, \quad b_3=0, \end{gathered} \tag{34}$$

a_3 arbitrary, b_4=0, and

$$0\cdot a_3+0=0, \quad b_4=0$$

so that a_3 is arbitrary.

All the a_j's ($j \neq 0$) and b_j's turn out to be zero so that there are only three (one less than the required number: 4) arbitrary parameters—s_0, b, and a_3. Consequently, the Laurent series (33) is not a valid local representation of the general solution of Eqs. (27) and (28) in the neighborhood of a movable singularity s_0. Therefore, Eqs. (27) and (28), and hence Eq. (23) does not possess the Painlevé property.

On the other hand, choosing a_3=0, one may verify that the Laurent series (33) indeed yields the exact solution

$$H=\frac{as}{(s-s_0)s_0}, \quad \theta=b. \tag{35}$$

In terms of the original variables, Eq. (35) gives a kink-shaped two-parameter solution

$$u=\sqrt{\frac{2(1-c^2)}{\mu}}\,\frac{1}{x_1-cx_2}\,e^{(i/4x_3)(x_2^2-x_1^2)+ib}, \tag{36}$$

which is localized in the x_1, x_2 plane about the line x_1-cx_2=0. Since μ<0 for nonlinear gravity waves and H was taken to be real, we require from Eq. (36) that $|c|$>1. The kink-shaped exact solution (36) is sketched in Fig. 1.

2. Symmetry group corresponding to the generator X_6

The symmetry group corresponding to the parameter γ and generated by the generator X_6 turns out to be peculiar to the deep-water case because it is not recovered by taking the deep-water limit of the symmetry groups associated with the finite-depth water case (see Sec. III). Let us consider first the similarity reductions of Eq. (4) associated with the symmetry group corresponding to the generator X_6.

The invariant solutions corresponding to X_6 satisfy

$$\frac{dx_1}{x_2}=\frac{dx_2}{x_1} \tag{37}$$

and so have the form

$$u(x_1,x_2,x_3)=u(x_3,x_4), \tag{38}$$

where

$$x_4=x_1^2-x_2^2. \tag{39}$$

Using Eq. (38), Eq. (4) becomes

$$iu_3-4u_4-4x_4^2u_{44}+\mu u^2u^*=0. \tag{40}$$

In order to see whether Eq. (40) can be reduced further to an ordinary differential equation, let us write it in the following form:

$$iu_3-u_4-x_4^2u_{44}+\mu u^2u^*=0 \tag{41}$$

and consider the invariance of Eq. (41) under a local Lie group of infinitesimal transformations of the form

$$\left.\begin{aligned}\bar{u}&=u+\epsilon\zeta(x_3,x_4,u,u^*)+O(\epsilon^2),\\ \bar{x}_j&=x_j+\epsilon\xi^{(j)}(x_3,x_4,u,u^*)+O(\epsilon^2),\quad j=3,4.\end{aligned}\right\} \tag{42}$$

This yields the following condition on $\xi^{(j)}$ and ζ (and its extensions $\zeta^{(k)}$):

$$i\zeta_3-\zeta_4^{(1)}-x_4^2\zeta_{44}^{(2)}-2x_4u_{44}\xi^{(4)}+2\mu uu^*\zeta+\mu u^2\zeta^*=0, \tag{43}$$

where the extensions $\zeta^{(k)}$ are given by expressions similar to those in Eq. (16).

Noting that $\xi^{(j)}$ and ζ satisfy

$$\frac{\partial\xi^{(j)}}{\partial u}=0,\quad j=3,4,\quad \frac{\partial^2\zeta}{\partial u^2}=0 \tag{44}$$

and following the standard procedure to obtain the symmetry group generators, we have

$$\xi^{(3)}=\sigma,\quad \xi^{(4)}=0,\quad \zeta=i\omega u. \tag{45}$$

The two-parameter symmetry group given by Eq. (45) is already contained in Eq. (37). As a result, one cannot introduce yet another similarity variable to reduce Eq. (40) to an ordinary differential equation. However, one may put

$$u=U(\phi)e^{i\lambda x_3}, \tag{46}$$

where $\phi=x_4\equiv x_1^2-x_2^2$ and reduce Eq. (40) to an ordinary differential equation

$$-\lambda U-U'-\phi^2U''+\mu U^2U^*=0, \tag{47}$$

where primes denote differentiation with respect to ϕ.

In order to investigate the Painlevé property of the solutions of Eq. (47), it is convenient to put

$$U=V(\phi)e^{i\theta(\phi)}, \tag{48}$$

where V and θ are real, and separate the resulting equation into real and imaginary parts

$$-\lambda V-V'-\phi^2V''+\phi^2V\theta'^2+\mu V^3=0, \tag{49}$$

$$-V\theta'-2\phi^2V'\theta'-\phi^2V\theta''=0. \tag{50}$$

The Painlevé property can be proven in a straightforward manner by expanding the solutions into a Laurent series and investigating whether these expansions contain sufficient number of arbitrary constants to cover the entire solution manifold of Eqs. (49) and (50).

The leading-order behavior of solutions of Eqs. (49) and (50) at a movable singularity ϕ_0 is determined by substituting

$$V=az^{\alpha_1}, \quad \theta=bz^{\alpha_2}, \quad z\equiv\phi-\phi_0 \tag{51}$$

into Eqs. (49) and (50), and balancing the most singular terms, to obtain

$$\left.\begin{array}{l} a^2=\dfrac{2\phi_0^2}{\mu}, \quad \alpha_1=-1, \quad \alpha_2=0, \\ b \quad \text{arbitrary.} \end{array}\right\} \tag{52}$$

The resonances are determined by setting

$$V=az^{-1}+pz^{-1+\sigma} \tag{53}$$

and balancing the most singular terms again in Eqs. (49) and (50). This gives

$$\sigma=-1,4 \tag{54}$$

in order that p is arbitrary.

We thus consider the expansions

$$\begin{aligned} V&=az^{-1}+a_0+a_1z+a_2z^2+a_3z^3+\cdots, \\ \theta&=b+b_1z+b_2z^2+b_3z^3+\cdots. \end{aligned} \tag{55}$$

Substituting Eq. (55) into Eqs. (49) and (50), and collecting terms of equal powers of z, we obtain for the a_j's and b_j's

$$\begin{gathered} a^2=\frac{2\phi_0^2}{\mu}, \quad b \quad \text{arbitrary}, \quad a_0=\frac{a(4\phi_0-1)}{6\phi_0^2}, \quad b_1=0, \\ a_1=\frac{(6\lambda-4)\phi_0^2+8\phi_0-1}{36\phi_0^4}a, \quad b_2=0, \\ a_2=-\frac{(18\lambda-8)\phi_0^3+(45-9\lambda)\phi_0^2-21\phi_0+2}{108\phi_0^6}a, \quad b_3 \quad \text{arbitrary}, \\ 4a_2(\phi_0-1)+2a_1+0\cdot a_3=0, \quad (\phi_0-1)b_3-b_4\phi_0^2=0. \end{gathered} \tag{56}$$

However, since

$$a_2 \neq -\frac{a_1}{2(\phi_0-1)} \quad \text{and} \quad \phi_0 \neq 1$$

it is obvious that the last two relations in (56) cannot be satisfied. This implies that a_3 cannot be arbitrary so that the Laurent series (55) is not a valid local representation of the general solution of Eqs. (49) and (50) in the neighborhood of a movable singularity ϕ_0. Therefore, like the similarity reductions of Eq. (4) associated with the symmetry group corresponding to the parameter α, Eqs. (49) and (50), and hence Eq. (41) do not possess the Painlevé property.

III. NONLINEAR EVOLUTION EQUATIONS FOR MODULATED GRAVITY WAVE TRAINS IN WATER OF FINITE DEPTH

In order to investigate the analytic properties of Eqs. (2) and (3), let us first rewrite them in the following form:

$$iu_3-u_{11}+u_{22}+\mu u^2u^*-\nu uv=0, \tag{57}$$

$$v_{11}+v_{22}+\nu(u_{11}u^*+2u_1u_1^*+uu_{11}^*)=0. \tag{58}$$

Here, μ and ν are real interaction parameters ($\mu<0$ and $\nu>0$ for nonlinear gravity waves).

A. Conservation laws

It may be readily verified that Eqs. (57) and (58) have a variational characterization. It corresponds to extremizing the action integral

$$J(u,\nu)=\int\int\int L(x_1,x_2,x_3,u_1,u_2,u_3,\psi_{11},\psi_{22})dx_1\,dx_2\,dx_3, \tag{59}$$

where $\psi_{11}\equiv v$ and the Lagrangian density L is given by

$$L=\frac{i}{2}(u^*u_3-uu_3^*)+|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4-\nu\psi_{11}|u|^2-\tfrac{1}{2}\psi_{11}^2-\tfrac{1}{2}\psi_{11}\psi_{22}. \tag{60}$$

Conservation laws can be found by applying Noether's theorem to the functional $J(u,\nu)$. $J(u,\nu)$ is invariant under translations in x_1, x_2, and x_3 and the scaling $u \Rightarrow ue^{i\epsilon}$. Thus, under a local Lie group of transformations

$$\bar{x}_\alpha=x_\alpha+\epsilon\xi^{(\alpha)}+O(\epsilon^2),\quad \bar{u}=u+\epsilon\eta+O(\epsilon^2),\quad \bar{v}=v+\epsilon\zeta+O(\epsilon^2) \tag{61}$$

defined by infinitesimal generators ξ^α, $\alpha=1,2,3$, η, and ζ, we have the following conservation laws (Logan[16]):

$$\begin{aligned}\frac{\partial}{\partial x_\alpha}\Bigg[L\xi^{(\alpha)}&+\left(\frac{\partial L}{\partial u_\alpha}-\frac{\partial}{\partial x_\beta}\frac{\partial L}{\partial u_{\beta\alpha}}\right)(\eta-u_\gamma\xi^{(\gamma)})+\left(\frac{\partial L}{\partial u_\alpha^*}-\frac{\partial}{\partial x_\beta}\frac{\partial L}{\partial u_{\beta\alpha}^*}\right)(\eta^*-u_\gamma^*\zeta^{(\gamma)})\\&+\left(\frac{\partial L}{\partial v_\alpha}-\frac{\partial}{\partial x_\beta}\frac{\partial L}{\partial v_{\beta\alpha}}\right)(\zeta-u_\gamma\xi^{(\gamma)})+\frac{\partial L}{\partial u_{\alpha\beta}}\frac{\partial}{\partial x_\beta}(\eta-u_\gamma\xi^{(\gamma)})+\frac{\partial L}{\partial u_{\alpha\beta}^*}\frac{\partial}{\partial x_\beta}(\eta^*-u_\gamma^*\xi^{(\gamma)})\\&+\frac{\partial L}{\partial v_{\alpha\beta}}\frac{\partial}{\partial x_\beta}(\zeta-u_\gamma\xi^{(\gamma)})\Bigg]=0,\end{aligned} \tag{62}$$

where repeated indices are summed over.

For the scaling in u, $\xi^{(\alpha)}=0$ for $\alpha=1,2,3$ and $\zeta=0$ and $\eta=iu$, and Eq. (62) becomes

$$\frac{\partial}{\partial x_1}(iu^*u_1-iuu_1^*)+\frac{\partial}{\partial x_2}(iuu_2^*-iu^*u_2)+\frac{\partial}{\partial x_3}(|u|^2)=0, \tag{63}$$

which shows that $|u|^2$ is an invariant.

For a translation in x_1, $\xi^{(1)}=1$ with $\xi^{(2)}=\xi^{(3)}=0$ and $\eta=\zeta=0$, and Eq. (62) becomes

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left(\frac{i}{2}(u^*u_3-uu_3^*)-|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4+\frac{1}{2}\psi_{11}^2+\frac{1}{2}\psi_{11}\psi_{22}\right.\\&\quad\left.-\frac{\partial}{\partial x_1}\left(\nu|u|^2+\psi_{11}+\frac{1}{2}\psi_{22}\right)\psi_1\right)+\frac{\partial}{\partial x_2}\left(u_1u_2^*+u_2u_1^*+\frac{1}{2}\psi_{11}\psi_{12}-\frac{1}{2}\psi_{112}\psi_1\right)\\&\quad+\frac{\partial}{\partial x_3}\left(\frac{i}{2}(uu_1^*-u^*u_1)\right)=0.\end{aligned} \tag{64}$$

Similarly, for a translation in x_2, $\xi^{(2)}=1$ with $\xi^{(1)}=\xi^{(3)}=0$ and $\eta=\zeta=0$, and Eq. (62) becomes

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left(-u_1^*u_2-u_1u_2^*+\left(\nu|u|^2+\psi_{11}+\frac{1}{2}\psi_{22}\right)\psi_{12}-\frac{\partial}{\partial x_1}\left(\nu|u|^2+\psi_{11}+\frac{1}{2}\psi_{22}\right)\psi_2\right)\\&\quad+\frac{\partial}{\partial x_2}\left(\frac{i}{2}(u^*u_3-uu_3^*)+|u_1|^2+|u_2|^2+\frac{\mu}{2}|u|^4-\nu|u|^2\psi_{11}-\frac{1}{2}\psi_{11}^2-\frac{1}{2}\psi_{112}\psi_2\right)\\&\quad+\frac{\partial}{\partial x_3}\left(\frac{i}{2}(uu_2^*-u^*u_2)\right)=0.\end{aligned} \tag{65}$$

For a translation in x_3, $\xi^{(3)}=1$ with $\xi^{(1)}=\xi^{(2)}=0$ and $\eta=\zeta=0$, and Eq. (62) becomes

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left(-u_1^*u_3-u_1u_3^*+\left(\nu|u|^2+\psi_{11}+\frac{1}{2}\psi_{22}\right)\psi_{13}-\frac{\partial}{\partial x_1}\left(\nu|u|^2+\psi_{11}+\frac{1}{2}\psi_{22}\right)\psi_3\right)\\&\quad+\frac{\partial}{\partial x_2}\left(u_2^*u_3+u_2u_3^*+\frac{1}{2}\psi_{11}\psi_{23}-\frac{1}{2}\psi_{70}\psi_3\right)+\frac{\partial}{\partial x_3}\left(|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4\right.\\&\quad\left.-\nu|u|^2\psi_{11}-\tfrac{1}{2}\psi_{11}^2-\tfrac{1}{2}\psi_{11}\psi_{22}\right)=0.\end{aligned} \tag{66}$$

Equation (66) leads to the "energy" invariant

$$\int\int\left(|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4-\nu|u|^2\psi_{11}-\frac{1}{2}\psi_{11}^2-\frac{1}{2}\psi_{11}\psi_{22}\right)dx_1\,dx_2=\text{const},$$

which, on using Eq. (58), may be reexpressed as

$$\int\int\left(|u_1|^2-|u_2|^2+\frac{\mu}{2}|u|^4+\frac{1}{2}\psi_{11}^2+\frac{1}{2}\psi_{12}^2\right)dx_1\,dx_2=\text{const}. \tag{67}$$

The last two terms representing the effect of finite depth in the above integral are positive definite, as the longitudinal dispersion term. Therefore, the effect of finite depth is to weaken the indefinite character of the "energy" and hence reduce the modulational instability of the gravity wave train.

B. The Lie group symmetries

Let us consider the invariance of Eqs. (57) and (58) under a local Lie group of infinitesimal transformations with generators $\xi^{(j)}$, $j=1,2,3$, and η and ζ

$$\begin{aligned}\bar{u}&=u+\epsilon\eta(x_1,x_2,x_3,u,u^*,v)+O(\epsilon^2),\\ \bar{v}&=v+\epsilon\zeta(x_1,x_2,x_3,u,u^*,v)+O(\epsilon^2),\\ \bar{x}_j&=x_j+\epsilon\xi^{(j)}(x_1,x_2,x_3,u,u^*,v)+O(\epsilon^2),\quad j=1,2,3.\end{aligned}\tag{68}$$

This then yields the following conditions on $\xi^{(j)}$, η, and ζ (and their extensions $\eta^{(k)}$ and $\zeta^{(k)}$):

$$i\eta_3^{(i)}-\eta_{11}^{(2)}+\eta_{22}^{(2)}+2\mu uu^*\eta+\mu u^2\eta^*-\nu v\eta-\nu u\zeta=0,\tag{69}$$

$$\zeta_{11}^{(2)}+\zeta_{22}^{(2)}+\nu(\eta_{11}^{(2)}u^*+2u_1^*\eta_1^{(1)}+2u_1\eta_1^{*(1)}+u_{11}\eta^*+u\eta_{11}^{*(2)}+u_{11}^*\eta)=0,\tag{70}$$

where the extensions $\eta^{(k)}$ and $\zeta^{(k)}$ are given by

$$\begin{aligned}\eta_j^{(1)}&=\eta_j+\eta_u u_j+\eta_v v_j-\xi_j^{(l)}u_l,\\ \eta_{11}^{(2)}&=\eta_{11}+2\eta_{1u}u_1+2\eta_{1v}v_1+\eta_u u_{11}+\eta_v v_{11}-\xi_{11}^{(l)}u_l-2\xi_1^{(l)}u_{1l},\\ \eta_{22}^{(2)}&=\eta_{22}+2\eta_{2u}u_2+2\eta_{2v}v_2+\eta_u u_{22}+\eta_v v_{22}-\xi_{22}^{(l)}u_l-2\xi_2^{(l)}u_{2l},\end{aligned}\tag{71}$$

etc. In Eq. (71), summation over repeated index l is implied.

Now, little is known about the form of the infinitesimal generators for systems of partial differential equations, like Eqs. (57) and (58), in contrast to the situation for a single partial differential equation (Bluman and Kumei[18]). We will seek to make some progress in the following by assuming that the infinitesimal generators $\xi^{(j)}$, η, and ζ satisfy the same conditions as those known for a single quasilinear-type partial differential equation

$$\begin{aligned}&\frac{\partial\xi^{(j)}}{\partial u}=0,\quad \frac{\partial\xi^{(j)}}{\partial v}=0;\quad j=1,2,3,\\ &\frac{\partial^2\eta}{\partial u^2}=0,\quad \frac{\partial^2\eta}{\partial u\partial v}=0,\quad \frac{\partial^2\eta}{\partial v^2}=0,\\ &\frac{\partial^2\zeta}{\partial u^2}=0,\quad \frac{\partial^2\zeta}{\partial u\partial v}=0,\quad \frac{\partial^2\zeta}{\partial v^2}=0.\end{aligned}\tag{72}$$

Using Eqs. (71) and (72), and following the standard procedure to obtain the symmetry group generators, we have

$$\xi^{(1)}=\alpha'(x_3)x_1+\hat{\alpha}(x_3), \quad \xi^{(2)}=\alpha'(x_3)x_2+\hat{\beta}(x_3), \quad \xi^{(3)}=2\alpha(x_3)+\hat{\gamma},$$

$$\eta=-\left[\frac{i\alpha''(x_3)}{4}(x_1^2-x_2^2)+\alpha'(x_3)+\frac{i\hat{\alpha}'(x_3)}{2}x_1-\frac{i\hat{\beta}'(x_3)}{2}x_2+i\omega\right]u, \tag{73}$$

$$\zeta=-2\alpha'(x_3)v+\frac{1}{\nu}\left[\frac{\alpha'''(x_3)}{4}(x_1^2-x_2^2)+\frac{\hat{\alpha}''(x_3)}{2}x_1-\frac{\hat{\beta}''(x_3)}{2}x_2\right].$$

Equation (73) gives an infinite parameter (α, $\hat{\alpha}$, and $\hat{\beta}$ being arbitrary functions) local Lie group of transformations admitted by Eqs. (57) and (58). It may be mentioned that the symmetry groups of the Davey–Stewartson equations were previously investigated by Tajiri and Hagiwara;[17] their results are only a subclass of the above set and correspond to $\hat{\alpha}(x_3)$ and $\hat{\beta}(x_3)$ being linear functions of x_3 and $\alpha(x_3)$ being a quadratic function of x_3. Champagne and Winternitz[20] investigated the symmetry groups of the Davey–Stewartson equations separately for the special case when the coupling parameter $\nu=1$; their results are also contained in the above set.

C. The Painlevé formulation

Consider the class of symmetry groups corresponding to the function $\alpha(x_3)$. The infinitesimal generators associated with this class of groups are given by

$$X_\alpha'=\alpha'(x_3)x_1\frac{\partial}{\partial x_1}+\alpha'(x_3)x_2\frac{\partial}{\partial x_2}+2\alpha(x_3)\frac{\partial}{\partial x_3}+\left[-\left\{\frac{i\alpha''(x_3)}{4}(x_1^2-x_2^2)+\alpha'(x_3)\right\}u\right]\frac{\partial}{\partial u}$$
$$+\left[-2\alpha'(x_3)v+\frac{1}{\nu}\left\{\frac{\alpha'''(x_3)}{4}(x_1^2-x_2^2)\right\}\right]\frac{\partial}{\partial v}. \tag{74}$$

The invariant solutions corresponding to X_α' satisfy

$$\frac{dx_1}{\alpha'(x_3)x_1}=\frac{dx_2}{\alpha'(x_3)x_2}=\frac{dx_3}{2\alpha(x_3)}=\frac{du}{-[(i\alpha''(x_3)/4)(x_1^2-x_2^2)+\alpha'(x_3)]u}$$
$$=\frac{dv}{-2\alpha'(x_3)v+(1/\nu)[(\alpha'''(x_3)/4)(x_1^2-x_2^2)]} \tag{75}$$

and so have the form

$$u=\alpha^{-1/2}e^{-(i\alpha'/8)(\psi^2-\chi^2)}F(\psi,\chi),$$
$$v=\frac{(\psi^2-\chi^2)}{8\nu\alpha}\left(\alpha\alpha''-\frac{1}{2}\alpha'^2\right)+\frac{1}{\alpha}G(\psi,\chi), \tag{76}$$

where

$$\psi=\frac{x_1}{\sqrt{\alpha}}, \quad \chi=\frac{x_2}{\sqrt{\alpha}}$$

and we have suppressed the argument on α, for compact notation.

Using Eq. (76), Eqs. (57) and (58) become

$$F_{\phi\phi}-F_{\psi\psi}+\mu F^2F^*-\nu FG=0, \tag{77}$$

$$G_{\phi\phi}+G_{\psi\psi}+\nu(F_{\psi\psi}F^*+2F_\psi F^*_\psi+FF^*_{\psi\psi})=0. \tag{78}$$

Equations (77) and (78) are the similarity-reductions of Eqs. (57) and (58) associated with the whole infinite-parameter symmetry group corresponding to the function α. The form of Eqs. (77) and (78) suggests a further scaling transformation

$$F=\frac{1}{\psi}\,g(s), \quad G=\frac{1}{\psi^2}\,h(s), \quad s=\frac{\psi}{\phi}. \tag{79}$$

Using Eq. (79), Eqs. (77) and (78) become

$$-\frac{2}{s^4}\,g+2\left(\frac{1}{s^3}+\frac{1}{s}\right)g'-\left(\frac{1}{s^2}-1\right)g''+\frac{\mu}{s^4}\,g^2g^*-\frac{\nu}{s^4}\,gh=0, \tag{80}$$

$$\left(\frac{1}{s^2}+1\right)h''+2\left(\frac{1}{s}-\frac{2}{s^3}\right)h'+\frac{6}{s^4}\,h+\nu\left[\frac{6}{s^4}\,|g|^2-\frac{4}{s^3}\,(g^*g'+gg^{*\prime})+\frac{2}{s^2}\,|g'|^2\right.$$
$$\left.+\frac{1}{s^2}\,(gg^{*\prime\prime}+g^*g'')\right]=0, \tag{81}$$

where primes denote differentiation with respect to s.

In order to investigate the Painlevé property of the solutions of Eqs. (80) and (81), it is convenient to put

$$g(s)=H(s)e^{i\theta(s)}. \tag{82}$$

Noting that h, H, and θ are real, we separate the resulting equations into real and imaginary parts

$$-2H+2(s+s^3)H'+(s^4-s^2)H''-(s^4-s^2)H\theta'^2+\mu H^3-\nu Hh=0, \tag{83}$$

$$2(1+s^2)H\theta'+2(s^3-s)H'\theta'+(s^3-s)H\theta''=0, \tag{84}$$

$$(s^2+s^4)h''+2(s^3-2s)h'+6h+\nu[6H^2-8sHH'+2s^2H'^2+2s^2HH'']=0. \tag{85}$$

The leading behavior of solutions of Eqs. (83)–(85) at a movable singularity s_0 is determined by substituting

$$H=az^{\alpha_1}, \quad \theta=bz^{\alpha_2}, \quad h=cz^{\alpha_3}, \quad z=s-s_0 \tag{86}$$

into Eqs. (83)–(85), and balancing the most singular terms, to obtain

$$\begin{aligned} a^2&=-\frac{2(s_0^4-s_0^2)}{\mu+[\nu^2/(1+s_0^2)]}, \quad \alpha_1=-1, \quad \alpha_2=0,\\ b&\ \text{arbitrary}, \quad c=-\frac{\nu a^2}{1+s_0^2}, \quad \alpha_3=-2. \end{aligned} \tag{87}$$

The resonances are next determined by setting

$$H=az^{-1}+pz^{-1+\sigma_1}, \quad h=cz^{-2}+qz^{-2+\sigma_2}, \quad \theta=b+rz^{\sigma_3} \tag{88}$$

and balancing the most singular terms again in Eqs. (83)–(85). This gives

$$\sigma_1=\sigma_2=-1,2,3,4; \quad \sigma_3=0,3 \tag{89}$$

in order that p, q, and r are arbitrary.

We thus consider the expansions

$$\begin{aligned} H&=az^{-1}+a_0+a_1z+a_2z^2+a_3z^3+\cdots, \\ \theta&=b+b_1z+b_2z^2+b_3z^3+b_4z^4+\cdots, \\ h&=cz^{-2}+c_{-1}z^{-1}+c_0+c_1z+c_2z^2+c_3z^3+\cdots. \end{aligned} \tag{90}$$

Substituting Eq. (90) into Eqs. (83)–(85), and collecting terms of equal powers of z, we obtain for the a_j's and b_j's

$$c=-\frac{\nu a^2}{1+s_0^2}, \quad a^2=-\frac{2(s_0^4-s_0^2)}{\mu+[\nu^2/(1+s_0^2)]}, \quad b \quad \text{arbitrary},$$

$$c_{-1}=-\frac{2\nu a^2}{s_0(1+s_0^2)}, \quad a_0=\frac{a}{s_0}, \quad b_1=0, \quad b_2=0, \quad b_3=0, \quad b_4=0,$$

$$0\cdot a_1=0, \quad 0\cdot a_2=0, \quad c_0=c_0(a_1), \quad c_1=c_1(a_1,a_2), \quad c_2=c_2(a_1,a_2), \quad a_3=a_3(a_1,a_2) \tag{91}$$

so that a_1 and a_2 are arbitrary.

Thus, there are only four (two less than the required number: 6) arbitrary parameters—s_0, b, a_1, and a_2. Consequently, the Laurent series (90) is not a valid local representation of the general solution of Eqs. (83)–(85) in the neighborhood of a movable singularity s_0. One may now be tempted to conclude that Eqs. (83)–(85), and hence, Eqs. (57) and (58) do not possess the Painlevé property because Eqs. (57) and (58) do not seem to admit another kind of singularity that is peculiar to the finite-depth case.

On the other hand, choosing $a_1=0$ and $a_2=0$, one finds that

$$a_3=0, \quad a_4=0,\ldots, \quad c_1=0, \quad c_2=0,\ldots$$

and hence the Laurent series (90) indeed yields the exact solution

$$H=\frac{as}{s_0(s-s_0)}, \quad \theta=b, \quad h=-\frac{\nu a^2s^2}{z^2s_0^2(1+s_0^2)}. \tag{92}$$

In terms of the original variables, Eq. (92) gives a kink-shaped infinite-parameter solution

$$u=\sqrt{\frac{2(1-c^2)}{\mu+[\nu^2/(1+c^2)]}}\,\frac{1}{x_1-cx_2}\,e^{(i\alpha'/8\alpha)(x_2^2-x_1^2)+ib}, \tag{93a}$$

$$v=-\frac{2\nu(1-c^2)}{\mu(1+c^2)+\nu^2}\,\frac{1}{(x_1-cx_2)^2}+\frac{x_1^2-x_2^2}{8\nu\alpha^2}\left(\alpha\alpha''-\frac{1}{2}\,\alpha'^2\right), \tag{93b}$$

which is localized in the x_1,x_2 plane about the line $x_1-cx_2=0$. Since $\mu<0$ for the nonlinear gravity wave and H was taken to be real, we require from Eq. (93) that $|c|\gtrsim 1$ when

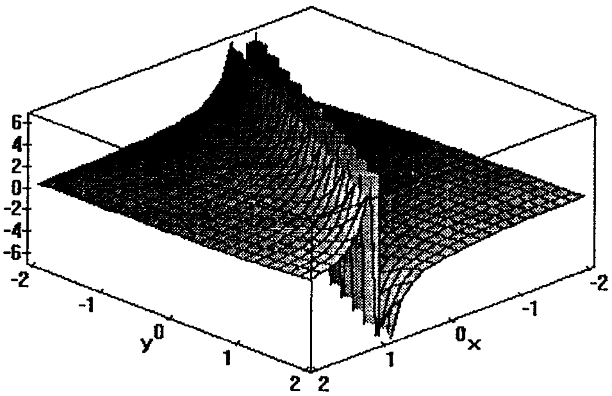

FIG. 2. Plot of $|u|$ as given in Eq. (93) with parameter values $b=0$, $c=1/2$, $\mu=-1$, and $\nu=5/2$.

$\mu+\nu^2/(1+c^2)\leq 0$. The kink-shaped exact solution (93) is sketched in Figs. 2 and 3 with $\alpha(x_3)=x_3^3$. Note that, in the limit $\nu\Rightarrow 0$, Eq. (93) completely reduces, however, to the corresponding exact solution (36) for the deep-water case.

D. Deep-water limit of the symmetry groups of the finite-depth water

In the deep-water limit $\nu\Rightarrow 0$, Eq. (76) implies that

$$\alpha\alpha''-\tfrac{1}{2}\alpha'^2=0 \tag{94}$$

in order that v is bounded. Equation (94) implies in turn, that

$$\alpha(x_3)=\sigma x_3^2+\kappa x_3+\Omega. \tag{95}$$

Thus, any departure of $\alpha(x_3)$ from Eq. (95) is peculiar to the finite-depth water case.

Observe that the symmetry groups persisting in the limit $\nu\Rightarrow 0$ are not the same as the symmetry groups existing when $\nu=0$. The symmetry group X_6 in Eq. (19), for the case $\nu=0$, is a case in point.

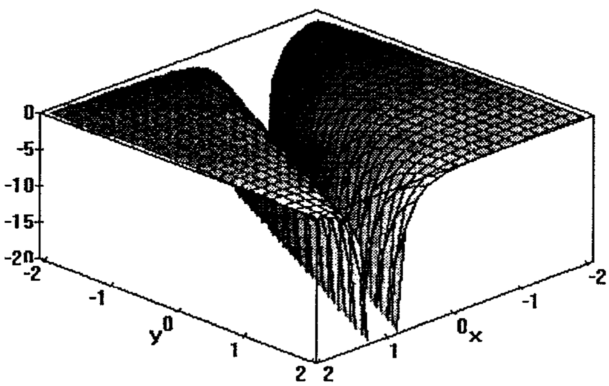

FIG. 3. Plot of v as given in Eq. (93) with parameter values $b=0$, $c=1/2$, $\mu=-1$, and $\nu=5/2$.

IV. DISCUSSION

We have seen in Secs. II and III that the group theoretical reductions of Eq. (1) and the system of Eqs. (2) and (3) for gravity wave trains to ordinary differential equations do not possess the Painlevé property. Indeed, Ramani[21] alluded to such a possibility previously. It may be noted that inverse-scattering transforms of these nonlinear evolution equations for general values of the parameters have not been given. Therefore, Eq. (1) does not appear to be integrable. However, it may be premature to make a similar conclusion with regard to Eqs. (2) and (3) because these equations may admit some other kind of singularity than the ones revealed by the group-theoretical reductions discussed in this article which may have the full dimensionality (i.e., possessing the right number of arbitrary parameters). One may search for such a fully dimensional singularity by dealing with Eqs. (2) and (3) directly, as in the approach proposed by Weiss, Tabor, and Carnevale.[22] This will be the subject of a forthcoming article.

The singularity, along $x_1=cx_2$, in the kink-shaped solutions (36) and (93) is not physical and simply implies that the basic assumptions that led to Eq. (1) and the system of Eqs. (2) and (3) have broken down near this line. The singular solutions (36) and (93) then become physically meaningful when they are used as "outer" solutions to be matched near $x_1=cx_2$ to smooth "inner" solutions of the full equations of water waves. This aspect will be addressed in a subsequent article.

On the other hand, for a typical finite-depth water case, like $\alpha(x_3)\sim x_3^3$, the exact solution (93) is identical to the one for the infinite-depth water case, Eq. (36), provided μ is suitably redefined. It is of interest to note that, a rederivation of the system (2) and (3) (see Appendix) shows that even the latter reduce to Eq. (1) if the coefficient of the cubic term μ is redefined suitably.

ACKNOWLEDGMENTS

The authors are thankful to Professor H. Segur, Professor A. Ramani, and Professor L. Redekopp for their helpful advice on these formulations.

APPENDIX

Consider an initially quiescent fluid subject to a gravitational field $-g\hat{i}_z$, and confined in the region $z<0$. Let us suppose that at time $t=0$, a progressive wave is established such that the elevation of the free surface is raised to $y=\eta$ where

$$t=0: \eta=a(\epsilon x,\epsilon y)e^{ikx}+\text{c.c.}, \quad \epsilon=ka\ll 1, \tag{A1}$$

which represents a sinusoidal form with slowly varying amplitude. The wave motions for subsequent times at the disturbed free surface described by $y=\eta(x,y,t;\epsilon)$ are governed by the following boundary-value problem:

$$-h<z<\eta: \quad \phi_{xx}+\phi_{yy}+\phi_{zz}=0, \tag{A2}$$

$$z=\eta: \quad \phi_z=\eta_t+\phi_x\eta_x+\phi_y\eta_y, \tag{A3}$$

$$\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)+g\eta=0, \tag{A4}$$

$$z=-h: \quad \phi_z=0, \tag{A5}$$

where ϕ denotes the perturbation in the velocity potential of the fluid. If the disturbance is assumed to be a progressive wave, we may introduce the following independent variables:

$$\xi=kx-\omega t, \quad \zeta=\epsilon(x-ct), \quad \mu=\epsilon y, \quad \tau=\epsilon^2 t, \tag{A6}$$

where c is the group velocity of the primary progressive wave. Equations (A2)–(A4) then become

$$-h<z<\eta:\quad k^2\phi_{\xi\xi}+\phi_{zz}+2\epsilon k\phi_{\xi\zeta}+\epsilon^2(\phi_{\zeta\zeta}+\phi_{\mu\mu})=0, \tag{A7}$$

$$z=\eta:\quad \phi_z=(k^2\phi_\xi-\omega)\eta_\xi+\epsilon((k\phi_\xi-c)\eta_\zeta+k\phi_\zeta\eta_\zeta)+\epsilon^2(\eta_\mu\phi_\mu+\eta_\tau+\phi_\zeta\eta_\zeta), \tag{A8}$$

$$-\omega\phi_\xi+\tfrac{1}{2}(k^2\phi_\xi^2+\phi_z^2)+g\eta+\epsilon(-c\phi_\zeta+k\phi_\xi\phi_\zeta)+\epsilon^2(\phi_\tau+\tfrac{1}{2}\phi_\zeta^2+\tfrac{1}{2}\phi_\mu^2)=0. \tag{A9}$$

Look for solutions of the form

$$\phi=\sum_1^\infty \epsilon^n\phi(\xi,z,\zeta,\mu,\tau)\quad\text{and}\quad \mu=\sum_1^\infty \epsilon^n\eta_n(\xi,\zeta,\mu,\tau). \tag{A10}$$

This leads to a hierarchy of problems of various orders in ϵ

$O(\epsilon)$:

$$-h<z<0:\quad k^2\phi_{1\xi\xi}+\phi_{1zz}=0, \tag{A11}$$

$$z=0:\quad \phi_{1z}+\omega\eta_{1\xi}=0, \tag{A12}$$

$$g\eta_1-\omega\phi_{1\xi}=0, \tag{A13}$$

$$z=-h:\quad \phi_{1z}=0. \tag{A14}$$

$O(\epsilon^2)$:

$$-h<z<0:\quad k^2\phi_{2\xi\xi}+\phi_{2zz}=-2k\phi_{1\xi\xi}, \tag{A15}$$

$$z=0:\quad \phi_{2z}+\omega\eta_{2\xi}=k^2\phi_{1\xi}\eta_{1\xi}-c\eta_{1\zeta}-\eta_1\phi_{1zz}, \tag{A16}$$

$$g\eta_2-\omega\phi_{2\xi}=-\tfrac{1}{2}k^2\phi_{1\xi}^2-\tfrac{1}{2}\phi_{1z}^2+c\phi_{1\zeta}+\omega\phi_{1\xi z}\eta_1, \tag{A17}$$

$$z=-h:\quad \phi_{2z}=0. \tag{A18}$$

$O(\epsilon^3)$:

$$-h<z<0:\quad k^2\phi_{3,\xi\xi}+\phi_{3,zz}=-2k\phi_{2,\xi\zeta}-\phi_{1,\zeta\zeta}-\phi_{1,\mu\mu}, \tag{A19}$$

$$\begin{aligned} z=0:\quad \phi_{3,z}+\omega\eta_{3,\xi}={}&k^2\phi_{2,\xi}\eta_{1,\xi}+k^2\phi_{1,\xi}\eta_{2,\xi}+k^2\phi_{1,\xi z}\eta_{1,\xi}\eta_1-c\eta_{2,\zeta}-\eta_1\phi_{2,zz}\\ &+k\phi_{1,\xi}\eta_{1,\zeta}+k\phi_{1,\zeta}\eta_{1,\xi}-\tfrac{1}{2}\phi_{1,zzz}\eta_1^2+\eta_{1,\tau}-\eta_2\phi_{1,zz}, \end{aligned} \tag{A20}$$

$$\begin{aligned} g\eta_3-\omega\phi_{3\xi}={}&-(k^2\phi_{1,\xi}\phi_{2,\xi}+k^2\phi_{1,\xi}\phi_{1,\xi z}\eta_1+\phi_{1,z}\phi_{2,z}+\phi_{1,z}\phi_{1,zz}\eta_1)+c\phi_{2,\zeta}+\omega\phi_{1,\xi z}\eta_2\\ &+\omega\phi_{2,\xi z}\eta_1+\tfrac{1}{2}\omega\phi_{1,\xi zz}\eta_1^2+c\phi_{1,\zeta z}\eta_1-k\phi_{1,\xi}\phi_{1,\zeta}-\phi_{1,\tau}, \end{aligned} \tag{A21}$$

$$z=-h:\quad \phi_{3,z}=0. \tag{A22}$$

From Eqs. (A11)–(A14), we obtain the linear results

$$\eta_1=A_1(\zeta,\mu,\tau)e^{i\xi}+\text{c.c.}, \tag{A23}$$

$$\phi_1=-\frac{g}{\omega}[iA_1(\zeta,\mu,\tau)e^{i\xi}+\text{c.c.}]\frac{\cosh k(z+h)}{\cosh kh}+\Phi(\zeta,\mu,\tau), \tag{A24}$$

$$\omega^2=gk\sigma, \quad \sigma=\tanh kh, \tag{A25}$$

where $\Phi(\zeta,\mu,\tau)$ represents the mean flow.

Using Eqs. (A23)–(A25), Eqs. (A15)–(A18) give

$$\phi_2=(A_{1\zeta}e^{i\xi}+\text{c.c.})\frac{(c+h\sigma\omega)\cosh k(z+h)}{k\sigma\cosh kh}+\left(\frac{3i\omega}{4\sigma^4}(\sigma^2-1)(1+\sigma^2)A_1^2e^{2i\xi}+\text{c.c.}\right)$$
$$\times\frac{\cosh 2k(z+h)}{\cosh 2kh}-\frac{g}{\omega}(A_{1\zeta}(\zeta,\mu,\tau)e^{i\xi}+\text{c.c.})\frac{(z+h)\sinh k(z+h)}{\cosh kh}, \tag{A26}$$

$$\eta_2=\frac{k}{2\sigma^3}(3-\sigma^2)A_1^2e^{2i\xi}+\text{c.c.}, \tag{A27}$$

$$c\Phi_\zeta=\frac{kg}{\sigma}(1-\sigma^2)(|A_1|^2)_\zeta. \tag{A28}$$

Using Eqs. (A23)–(A28), Eqs. (A19) and (A22) give

$$\phi_3=-\frac{1}{2}(z+h)^2(\Phi_{\zeta\zeta}+\Phi_{\mu\mu})+\frac{ig}{2k\omega}(A_{1\mu\mu}-A_{1\zeta\zeta})\frac{(z+h)\sinh k(z+h)}{\cosh kh}e^{i\xi}$$
$$-\frac{i(c+h\sigma\omega)}{k\sigma}A_{1\zeta\zeta}e^{i\xi}\frac{(z+h)\sinh k(z+h)}{\cosh kh}+\frac{ig}{2\omega}A_{1\zeta\zeta}e^{i\xi}\frac{(z+h)^2\cosh k(z+h)}{\cosh kh}$$
$$+Fe^{i\xi}\frac{\cosh k(z+h)}{\cosh kh}+\text{c.c.}, \tag{A29}$$

$$\eta_3=He^{i\xi}+\text{c.c.}, \tag{A30}$$

where $F(\zeta,\mu,\tau)$ and $H(\zeta,\mu,\tau)$ are determined from the boundary conditions (A20) and (A21). Using Eqs. (A23)–(A30), Eqs. (A20), and (A21) give

$$gh(\Phi_{\zeta\zeta}+\Phi_{\mu\mu})=\frac{-2g^2k}{\omega}(|A|^2)_\zeta, \tag{A31}$$

$$F=\frac{-ig}{2k^2\omega\sigma}A_{1\mu\mu}(\sigma+hk)+\frac{1}{k\sigma}A_{1\tau}+\frac{i\omega}{2k^2\sigma}\left\{\frac{1}{k}+\frac{h}{\sigma}\left(\frac{kh}{\sigma}+\sigma^2\right)\right\}A_{1\zeta\zeta}-\frac{ig}{2k\sigma\omega}(2h+h^2k\sigma)A_{1\zeta\zeta}$$
$$-\frac{3ik\omega}{2\sigma^5}(\sigma^2-1)(1+\sigma^2)A_1^2A_1^*+\frac{igk^2}{2\omega\sigma^4}(3-\sigma^2)A_1^2A_1^*+\frac{3igk^2}{2\omega}A_1^2A_1^*+\frac{i}{\sigma}\Phi_\zeta A_1, \tag{A32}$$

$$-i\omega F=\frac{-gh\sigma}{2k}A_{1\mu\mu}+\frac{ig}{\omega}A_{1\tau}+\left\{\frac{c^2+h\sigma\omega c}{k\sigma}-\frac{cgh\sigma}{\omega}+\frac{\omega h(c+h\omega\sigma)}{k}-\frac{gh^2}{2}\right\}A_{1\zeta\zeta}+\left\{(\sigma^2-1)\right.$$
$$\left.\times(1+\sigma^2)\left\{\frac{3gk^2}{2\sigma^4}-\frac{3gk^2\sigma}{\sigma^3(1+\sigma^2)}\right\}-\frac{5}{2}k^2g+\frac{k^2g}{2\sigma^2}(3-\sigma^2)\right\}A_1^2A_1^*-\frac{kg}{\omega}\Phi_\zeta A_1. \tag{A33}$$

Eliminating F from Eqs. (A32) and (A33), we obtain

$$2i\omega A_{1\tau}+cc_pA_{1\mu\mu}+\omega\omega''A_{1\zeta\zeta}=2\omega k(1+(1-\sigma^2)c/2c_p)A_1\Phi_\zeta+\frac{k^2\omega^2}{2\sigma^2}\times\left\{\frac{(1-\sigma^2)(9-\sigma^2)}{\sigma^2}+8\sigma^2-2(1-\sigma^2)^2\right\}A_1^2A_1^*. \tag{A34}$$

On using Eq. (A28), Eq. (A31) becomes

$$(gh-c^2)\Phi_{\zeta\zeta}+gh\Phi_{\mu\mu}=\frac{-g^2k}{\omega}(2+(1-\sigma^2)c/c_p)(|A_1|^2)_\zeta. \tag{A35}$$

Here, $c_p\equiv\omega/k$, and the prime denotes differentiation with respect to k. Equations (A34) and (A35) are the Davey–Stewartson equations.

On the other hand, if we use Eq. (A28) again, Eq. (A34) can be rewritten as

$$2i\omega A_{1\tau}+cc_pA_{1\mu\mu}+\omega\omega''A_{1\zeta\zeta}=\frac{k^2\omega^2}{2\sigma^2}\left\{\frac{(1-\sigma^2)(9-\sigma^2)}{\sigma^2}+8\sigma^2+\frac{4c_p}{c}(1-\sigma^2)\right\}A_1^2A_1^*, \tag{A36}$$

which is the same as the one for the infinite-depth water case, provided the coefficient of the cubic nonlinear term on the right hand side is suitably redefined!

[1] V. E. Zakharov, Sov. Phys. JETP **35**, 908 (1972).
[2] D. J. Benney and G. Roskes, Stud. Appl. Math. **48**, 377 (1969).
[3] H. C. Yuen and W. E. Fergusson, Phys. Fluids **21**, 2116 (1978).
[4] D. V. Martin and H. C. Yuen, Phys. Fluids **23**, 881 (1980).
[5] B. K. Shivamoggi and R. N. Mohapatra, Phys. Lett. A **131**, 265 (1988).
[6] A. Thyagaraja, Phys. Fluids **22**, 2093 (1979).
[7] A. Davey and K. Stewartson, Proc. R. Soc. London, Ser. A **338**, 377 (1974).
[8] V. D. Djordjevic and L. G. Redekopp, J. Fluid Mech. **79**, 703 (1977).
[9] N. E. Freeman and A. Davey, Proc. R. Soc. London, Ser. A **344**, 427 (1975).
[10] D. Anker and N. E. Freeman, Proc. R. Soc. London, Ser. A **360**, 520 (1978).
[11] M. J. Ablowitz and R. Haberman, Phys. Rev. Lett. **35**, 1185 (1975).
[12] M. J. Ablowitz and H. Segur, J. Fluid Mech. **92**, 691 (1979).
[13] M. Tajiri and M. Hagiwara, J. Phys. Soc. Jpn. **52**, 3727 (1983).
[14] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[15] M. J. Ablowitz and H. Segur, *Solitons and the Inverse-Scattering Transform* (SIAM Publications, Philadelphia, 1981).
[16] J. D. Logan, *Invariant Variational Principles* (Academic, New York, 1977).
[17] R. A. Cairns, J. Fluid Mech. **92**, 1 (1979).
[18] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer-Verlag, New York, 1989).
[19] M. Tajiri, J. Phys. Soc. Jpn. **52**, 1908 (1983).
[20] B. Champagne and P. Winternitz, J. Math. Phys. **29**, 1 (1988).
[21] A. Ramani, Ph.D. thesis, Princeton University, 1979.
[22] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24**, 522 (1983).